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One-loop four-point function in non-commutative $\mathcal{N} = 4$ Yang-Mills theory

Alberto Santambrogio and Daniela Zanon

Dipartimento di Fisica dell'Università di Milano and INFN
Sezione di Milano, Via Celoria 16, 20133 Milano, Italy
E-mail: alberto.santambrogio@mi.infn.it,
daniela.zanon@mi.infn.it

ABSTRACT: We compute the one-loop four-point function in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group U(N). We perform the calculation in $\mathcal{N} = 1$ superspace using the background field method and obtain the complete off-shell contributions to the effective action from planar and non-planar supergraphs. In the low-energy approximation the result simplifies and we can study its properties under gauge transformations. It appears that the non-planar contributions do not maintain the gauge invariance of the classical action.

KEYWORDS: Extended Supersymmetry, Superspaces, Non-Commutative Geometry.

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1. Introduction

The realization that the field theory limit of open strings in the presence of a constant B-field gives rise to non-commutative gauge theories has driven much attention on the subject [1, 2, 3]. Several computations have been performed in the string amplitude framework [4] as well as in the perturbative field theory one [5]. In dealing with supersymmetric theories it is often advantageous to use a superspace formulation. Such an approach has been introduced also for non-commutative theories [6]–[9]. The net result is that one constructs the theory in terms of standard superfields and implements the non commutativity through the $*$-product which replaces the ordinary multiplication. Since the $*$-product does not affect the fermionic coordinates, the quantization and the derivation of the Feynman rules can be performed following the same procedure as for commutative theories. Exponential factors in the interactions, stemming from the $*$-products between the superfields, are essentially the only new feature which distinguishes the non-commutative perturbation theory from the commutative one. Indeed it has been shown that one can use supergraph techniques and standard D-algebra both for chiral matter models [8] and for super gauge theories [9].

In this paper we continue the work presented in [9]. There the non-commutative $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group $\mathrm{U}(N)$ has been studied in $\mathcal{N}=1$ superspace and the background field quantization has been derived for one-loop calculations in perturbation theory. Here we follow the ideas and techniques introduced in [9] and compute the complete four-point function with external gauge fields at one loop. We analyze the various contributions to the effective action from planar and non-planar supergraphs. We find that our result is in accordance with the off-shell extension of the result from one-loop four gluon scattering on parallel D3-branes [10]. As noticed in [10], while the on-shell S-matrix calculation is gauge invariant, the same is not true for the off-shell contribution to the effective action. The terms that do not respect the non-commutative gauge invariance of the classical action are the ones produced by non-planar diagram contributions. At the end of the paper we discuss this issue.

Section 2 contains a brief introduction to the background field method and the derivation of the Feynman rules relevant for one-loop calculations. We also give details on how we organize the Wick expansion and pay special attention to the various steps necessary for the construction of the non-planar diagrams. In sections 3 and 4 we give the results for the planar and non-planar contributions. In section 5 we study the four-point function in the low-energy approximation. Finally we conclude with comments on the apparent loss of gauge invariance of the one-loop result and on possible attempts to resolve the problem.

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2. Background covariant quantization for non-commutative $\mathcal{N}=4$ Yang-Mills

The $*$-product is the object of primary interest in non-commutative field theories. Such an operation has been introduced also in the context of non-commutative supersymmetric theories. It acts as a non-local multiplication for chiral and gauge superfields in the following way:

$$(\phi_1 * \phi_2)(x,\theta,\bar\theta) \equiv e^{\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}\,\phi_1(x,\theta,\bar\theta)\phi_2(y,\theta,\bar\theta)|_{y=x}\,. \tag{2.1}$$

It maintains explicit supersymmetry and it can be used consistently in the construction of non-commutative supersymmetric actions.

In terms of $\mathcal{N}=1$ superfields the classical action of the non-commutative $\mathcal{N}=4$ supersymmetric Yang-Mills theory can be written as (we use the notations and conventions adopted in [9, 11])

$$\begin{aligned} S = \frac{1}{g^2}\,\mathrm{Tr}\Bigg(\int d^4x\, d^4\theta\, e^{-V}\bar\Phi_i e^V\Phi^i + \frac{1}{2}\int d^4x\, d^2\theta\, W^2 + \frac{1}{2}\int d^4x\, d^2\bar\theta\, \bar W^2 + \\ + \frac{1}{3!}\int d^4x\, d^2\theta\, i\epsilon_{ijk}\Phi^i[\Phi^j,\Phi^k] + \frac{1}{3!}\int d^4x\, d^2\bar\theta\, i\epsilon^{ijk}\bar\Phi_i[\bar\Phi_j,\bar\Phi_k]\Bigg)\Bigg|_*\,, \end{aligned} \tag{2.2}$$

where the symbol $|_*$ denotes multiplication as defined in (2.1). In (2.2) the Φ^i with

$i = 1,2,3$ are three chiral superfields, while $W^\alpha = i\bar{D}^2(e_*^{-V} * D^\alpha e_*^V)$ is the gauge superfield strength. All the fields are Lie-algebra valued, e.g. $\Phi^i = \Phi_a^i T_a$, in the adjoint representation of $\mathrm{U}(N)$.

The background field quantization has been used efficiently in perturbative calculations for commutative SYM theory [11, 12, 13]. In [9] it has been shown that the superspace background field method can be applied in the quantization procedure even for non-commutative gauge theories. This approach simplifies the calculations dramatically. At one loop the four-point vector amplitude receives contributions only from the vector superfields themselves, since the ghost loops exactly cancel the chiral matter loops. Moreover, as in ordinary commutative gauge theories, the background field method allows to express the one-loop corrections to the action in (2.2) in terms of field strengths.

The background field quantization can be implemented in the non-commutative theory essentially because the $*$-product does not affect the superspace fermionic coordinates and therefore does not alter the various properties of the superfields. In all the steps it is sufficient to introduce the appropriate $*$-multiplication [9]. Here we briefly summarize how one performs the quantum background splitting. One defines covariant derivatives

$$\nabla_\alpha = e_*^{-V/2} * D_\alpha \, e_*^{V/2} \,, \qquad \bar{\nabla}_{\dot{\alpha}} = e_*^{V/2} * \bar{D}_{\dot{\alpha}} \, e_*^{-V/2} \tag{2.3}$$

so that the gauge lagrangian becomes

$$\frac{1}{2} \mathrm{Tr} \;\; W^\alpha * W_\alpha = - \, \mathrm{Tr} \left(\frac{1}{2} \left[\bar{\nabla}^{\dot{\alpha}}, \{ \bar{\nabla}_{\dot{\alpha}}, \nabla_\alpha \} \right] \right)^2 . \tag{2.4}$$

The splitting is obtained via the introduction of a quantum prepotential V and background covariant derivatives

$$\nabla_\alpha \longrightarrow e_*^{-V/2} * \nabla_\alpha e_*^{V/2} \,, \qquad \bar{\nabla}_{\dot{\alpha}} \longrightarrow e_*^{V/2} * \bar{\nabla}_{\dot{\alpha}} e_*^{-V/2} \,. \tag{2.5}$$

On the r.h.s. of (2.5) the covariant derivatives are expressed in terms of background connections, i.e.

$$\nabla_\alpha = D_\alpha - i\mathbf{\Gamma}_\alpha \,, \qquad \bar{\nabla}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} - i\bar{\mathbf{\Gamma}}_{\dot{\alpha}} \,, \qquad \nabla_a = \partial_a - i\mathbf{\Gamma}_a \,. \tag{2.6}$$

Adding to the classical lagrangian (2.4) background covariantly chiral gauge fixing functions, $\nabla^2 V$ and $\bar{\nabla}^2 V$. one obtains

$$-\frac{1}{2g^2} \mathrm{Tr} \left[\left(e_*^{-V} * \nabla^\alpha e_*^V \right) * \bar{\nabla}^2 \left(e_*^{-V} * \nabla_\alpha e_*^V \right) + V * \left(\nabla^2 \bar{\nabla}^2 + \bar{\nabla}^2 \nabla^2 \right) V \right] . \tag{2.7}$$

We are interested in one-loop calculations, therefore we can restrict our attention to the terms in the action which are quadratic in the quantum fields. With the

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connections defined in (2.6) we obtain

$$-\frac{1}{2g^2}\,\mathrm{Tr}\,V*\left[\frac{1}{2}\partial^a\partial_a - i\mathbf{\Gamma}^a*\partial_a - \frac{i}{2}\partial^a\mathbf{\Gamma}_a* - \frac{1}{2}\mathbf{\Gamma}^a*\mathbf{\Gamma}_a* - i\mathbf{W}^\alpha*(D_\alpha - i\mathbf{\Gamma}_\alpha*) - \right.$$
$$\left. - i\bar{\mathbf{W}}^{\dot\alpha}*(\bar{D}_{\dot\alpha} - i\bar{\mathbf{\Gamma}}_{\dot\alpha}*)\right]V\,. \tag{2.8}$$

We have interactions with the background fields at most linear in the D's. Since superspace Feynman rules require the presence of two D's and two $\bar{D}$'s for a non-zero loop contribution, at least four vertices are needed and the first non-vanishing correction to the effective action is at the level of the four-point function. Thus following [9], we consider

$$-\frac{1}{2g^2}\,\mathrm{Tr}\,V*\left[\frac{1}{2}\partial^a\partial_a - i\mathbf{W}^\alpha*D_\alpha - i\bar{\mathbf{W}}^{\dot\alpha}*\bar{D}_{\dot\alpha}\right]V \tag{2.9}$$

and read from here the superspace Feynman rules which, with the appropriate $*$-product inserted, are the standard ones. In momentum space the vector propagators are

$$\left\langle V^a(\theta)V^b(\theta')\right\rangle = -\frac{g^2}{p^2}\delta^{ab}\delta^4(\theta-\theta') \tag{2.10}$$

and the relevant interactions with the background are given in (2.9). As already discussed in [9], one-loop contributions with chiral matter fields and ghosts inside the loop cancel among themselves, so we will consider only quantum vector loops.

In momentum space, with momenta k_1, k_2, k_3 flowing into the vertex ($k_1+k_2+k_3=0$), the three-point interactions can be written as

$$\frac{1}{2g^2}\left(\mathcal{U}(k_1,k_2,k_3)+\bar{\mathcal{U}}(k_1,k_2,k_3)\right) \equiv \frac{1}{2g^2}V_a(k_1)\left[i\mathbf{W}^\alpha_b(k_2)D_\alpha + i\bar{\mathbf{W}}^{\dot\alpha}_b(k_2)\bar{D}_{\dot\alpha}\right]\times$$
$$\times V_c(k_3)\,\mathrm{Tr}\left(T^aT^bT^c\right)\times$$
$$\times\, e^{-\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)}\,, \tag{2.11}$$

where $k_i\times k_j \equiv (k_i)_\mu\Theta^{\mu\nu}(k_j)_\nu$. Inserting a vertex in the loop we obtain two types of terms, an *untwisted* and a *twisted* term, i.e.

$$\mathcal{U}(k_1,k_2,k_3) \longrightarrow V_a(k_1)i\mathbf{W}^\alpha_b(k_2)D_\alpha V_c(k_3)\left[\mathrm{Tr}(T^aT^bT^c)e^{-\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)} - \right.$$
$$- \mathrm{Tr}(T^cT^bT^a)\times$$
$$\left.\times\, e^{\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)}\right]. \tag{2.12}$$

Now the quantum V lines have to be Wick contracted in the consecutive order in which they appear. As already mentioned, superspace D-algebra rules require two D and two $\bar{D}$ spinor derivatives in the loop in order to obtain a non-zero result and then

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the first contribution to the effective action is from the four-point function. From the Wick expansion we look for the forth order contribution

$$\frac{1}{4!(2g^2)^4}\,(\mathcal{U}+\bar{\mathcal{U}})^4 \tag{2.13}$$

and from (2.13) we select the terms containing two $\mathcal{U}$'s and two $\bar{\mathcal{U}}$'s.

There are two possible arrangements of the vertices in the loop, i.e. $\mathcal{U}\mathcal{U}\bar{\mathcal{U}}\bar{\mathcal{U}}$ and $\mathcal{U}\bar{\mathcal{U}}\mathcal{U}\bar{\mathcal{U}}$. For both cases the D-algebra is straightforward. It gives $D_\alpha D_\beta \bar{D}_{\dot\alpha}\bar{D}_{\dot\beta} \to C_{\beta\alpha}C_{\dot\beta\dot\alpha}$ and $D_\alpha \bar{D}_{\dot\alpha} D_\beta \bar{D}_{\dot\beta} \to -C_{\beta\alpha}C_{\dot\beta\dot\alpha}$, respectively for the two arrangements. After D-algebra we can write the vertices symbolically as (see (2.12))

$$\begin{aligned}\mathcal{U}^\alpha(k_1,k_2,k_3) &\equiv \mathcal{U}^\alpha_P(k_1,k_2,k_3)+\mathcal{U}^\alpha_T(k_1,k_2,k_3)\\ &\equiv V_a(k_1)i\mathbf{W}^\alpha_b(k_2)V_c(k_3)\Big[\mathrm{Tr}(T^aT^bT^c)\,e^{-\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)}-\\ &\qquad - \mathrm{Tr}(T^cT^bT^a)\,e^{\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)}\Big]\end{aligned} \tag{2.14}$$

again with the V quantum lines to be contracted in the order in which they appear in the loop. We have found it convenient to write the external background fields in the most symmetric way, that is

$$\begin{aligned}\mathcal{T}(1a,2b,3c,4d) = &\ \mathbf{W}^{\alpha a}(p_1)\mathbf{W}^b_\alpha(p_2)\bar{\mathbf{W}}^{\dot\alpha c}(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4)+\\ &+\mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\bar{\mathbf{W}}^c_{\dot\alpha}(p_3)\mathbf{W}^d_\alpha(p_4)-\\ &-\mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\mathbf{W}^c_\alpha(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4)+\text{h.c.}\,,\end{aligned} \tag{2.15}$$

where the minus sign in the third term takes into account the result from the D-algebra. The above expression is completely symmetric in the exchanges of any couple $(1a)\leftrightarrow(2b)\leftrightarrow(3c)\leftrightarrow(4d)$, a property that we will use extensively in order to write the final result in a simple form. Moreover from (2.15) one obtains the corresponding bosonic expression in a form which is directly comparable with the result from string amplitude calculations

$$\begin{aligned}&\int d^2\theta d^2\bar\theta\Big[\mathbf{W}^{\alpha a}(p_1)\mathbf{W}^b_\alpha(p_2)\bar{\mathbf{W}}^{\dot\alpha c}(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4)+\mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\bar{\mathbf{W}}^c_{\dot\alpha}(p_3)\mathbf{W}^d_\alpha(p_4)-\\ &\qquad -\mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\mathbf{W}^c_\alpha(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4)+\text{h.c.}\Big]\\ &\quad\longrightarrow t^{\alpha\beta\gamma\delta\mu\nu\rho\sigma}F^a_{\alpha\beta}(p_1)F^b_{\gamma\delta}(p_2)F^c_{\mu\nu}(p_3)F^d_{\rho\sigma}(p_4)\,,\end{aligned} \tag{2.16}$$

where $t^{\alpha\beta\gamma\delta\mu\nu\rho\sigma}$ is the symmetric tensor given e.g. in [15, formula (9.A.18)].

With all of this in mind the box diagram can be written as

$$\Big[\mathcal{U}^\alpha\,\mathcal{U}_\alpha\,\bar{\mathcal{U}}^{\dot\alpha}\,\bar{\mathcal{U}}_{\dot\alpha}+\mathcal{U}^\alpha\,\bar{\mathcal{U}}^{\dot\alpha}\,\bar{\mathcal{U}}_{\dot\alpha}\,\mathcal{U}_\alpha\,-\mathcal{U}^\alpha\,\bar{\mathcal{U}}^{\dot\alpha}\,\mathcal{U}_\alpha\,\bar{\mathcal{U}}_{\dot\alpha}+\text{h.c.}\Big] \tag{2.17}$$

with $\mathcal{U}^\alpha$ given in (2.14). Each term in (2.17) produces sixteen contributions: two of

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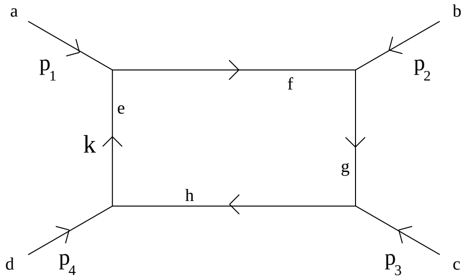

Figure 1: box diagram.

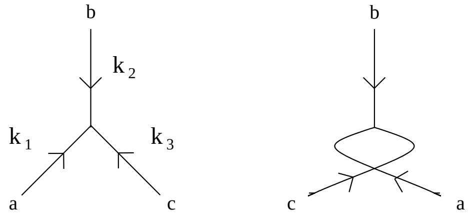

Figure 2: untwisted and twisted vertices.

them, i.e. the ones which contain all untwisted P and all twisted T vertices correspond to planar diagrams. All the others, i.e. the ones with two P and two T vertices (a total of six), the ones with one P and three T's (a total of four) and the ones with one T and three P's (a total of four), correspond to non-planar graphs. In any case contraction of the quantum V's produces four scalar propagators as in (2.10). If we call k the loop momentum and p_1, p_2, p_3, p_4 the external momenta as shown in figure 1, we have

$$\begin{aligned} I_0(k;p_1,\ldots,p_4) &= \frac{1}{(k+p_1)^2k^2(k-p_4)^2(k+p_1+p_2)^2} \\ &= \int_0^\infty \prod_{i=1}^4 d\alpha_i \, e^{-\alpha\{k+\frac{1}{\alpha}[\alpha_1 p_1-\alpha_3 p_4+\alpha_4(p_1+p_2)]\}^2} \times \\ &\qquad\qquad \times \; e^{-[\alpha_1 p_1^2+\alpha_3 p_4^2+\alpha_4(p_1+p_2)^2]} e^{\frac{1}{\alpha}[\alpha_1 p_1-\alpha_3 p_4+\alpha_4(p_1+p_2)]^2}\,, \end{aligned} \tag{2.18}$$

where $\alpha = \alpha_1+\alpha_2+\alpha_3+\alpha_4$. The vertices can be either of the *untwisted* P or of the *twisted* T type. With a labeling as in figure 2, we have

$$\begin{aligned} P &\longrightarrow \mathrm{Tr}(T^aT^bT^c)\, e^{-\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)} = \mathrm{Tr}(T^aT^bT^c)\, e^{-\frac{i}{2}k_2\times k_3}\,, \\ T &\longrightarrow -\,\mathrm{Tr}(T^cT^bT^a)\, e^{\frac{i}{2}(k_1\times k_2+k_2\times k_3+k_1\times k_3)} = -\,\mathrm{Tr}(T^cT^bT^a)\, e^{\frac{i}{2}k_2\times k_3}\,, \end{aligned} \tag{2.19}$$

where we have used $k_1+k_2+k_3=0$. For the diagram in figure 1 we have two of the following choices at each vertex:

$$\begin{aligned} P_1 &= \mathrm{Tr}(T_eT_aT_f)\, e^{\frac{i}{2}p_1\times k} &&\text{or}\qquad T_1 = -\,\mathrm{Tr}(T_fT_aT_e)\, e^{-\frac{i}{2}p_1\times k}\,, \\ P_2 &= \mathrm{Tr}(T_fT_bT_g)\, e^{\frac{i}{2}p_2\times(k+p_1)} &&\text{or}\qquad T_2 = -\,\mathrm{Tr}(T_gT_bT_f)\, e^{-\frac{i}{2}p_2\times(k+p_1)}\,, \\ P_3 &= \mathrm{Tr}(T_gT_cT_h)\, e^{\frac{i}{2}p_3\times(k-p_4)} &&\text{or}\qquad T_3 = -\,\mathrm{Tr}(T_hT_cT_g)\, e^{-\frac{i}{2}p_3\times(k-p_4)}\,, \\ P_4 &= \mathrm{Tr}(T_hT_dT_e)\, e^{\frac{i}{2}p_4\times k} &&\text{or}\qquad T_4 = -\,\mathrm{Tr}(T_eT_dT_h)\, e^{-\frac{i}{2}p_4\times k}\,. \end{aligned} \tag{2.20}$$

In the following two sections we will compute the full contribution to the effective action arising from all the possible combinations of untwisted and twisted vertices.

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3. Planar contributions

The planar diagrams correspond to terms which contain either four untwisted P vertices or four twisted T vertices. In these cases the internal momentum k cancels from the exponential factors in (2.20), and the remaining dependence from the external momenta is exactly such to reconstruct the $*$-product among the $\mathbf{W}$'s [9]. By using the relation $T^a_{ij}T^a_{kl} = \delta_{il}\delta_{jk}$ for the U(N) gauge matrices, we obtain the following contribution to the effective action [9]:

$$\begin{aligned}\Gamma_{\text{planar}} = \frac{N}{128}\,\mathrm{Tr}\int d^2\theta\, d^2\bar\theta\, \frac{d^4p_1\, d^4p_2\, d^4p_3\, d^4p_4}{(2\pi)^{16}}\,\delta\Big(\sum p_i\Big)\,\times \\ \times\int d^4k\, I_0(k;p_1,\ldots,p_4)\Big(\mathbf{W}^\alpha(p_1) * \mathbf{W}_\alpha(p_2) * \bar{\mathbf{W}}^{\dot\alpha}(p_3) * \bar{\mathbf{W}}_{\dot\alpha}(p_4) + \\ + \mathbf{W}^\alpha(p_1) * \bar{\mathbf{W}}^{\dot\alpha}(p_2) * \bar{\mathbf{W}}_{\dot\alpha}(p_3) * \mathbf{W}_\alpha(p_4) - \\ - \mathbf{W}^\alpha(p_1) * \bar{\mathbf{W}}^{\dot\alpha}(p_2) * \mathbf{W}_\alpha(p_3) * \bar{\mathbf{W}}_{\dot\alpha}(p_4) + \text{h.c.}\Big)\end{aligned} \tag{3.1}$$

with I_0 defined in (2.18). Making use of the relation in (2.16) one obtains the corresponding bosonic expression. The result in (3.1) is no surprise: it corresponds to the result in the commutative theory with ordinary products replaced by $*$-products.

4. Non-planar contributions

Now we focus on the analysis of non-planar contributions which represent the real novelty as compared to standard perturbative calculations. Indeed the phases from the $*$-products at the vertices end up containing a dependence on the loop momentum; this fact drastically changes the behaviour of the loop integral [5].

The various non-planar diagrams group themselves into two classes [9], graphs with two twisted vertices and graphs with one (or equivalently three) twisted vertex. For the first type of diagrams the trace on the U(N) matrices gives a factor like $\mathrm{Tr}(T^pT^q)\,\mathrm{Tr}(T^rT^s)$, while for the second type it gives $\mathrm{Tr}(T^p)\,\mathrm{Tr}(T^qT^rT^s)$. The total result can be written in the following form:

$$\begin{aligned}\Gamma_{\text{nonplanar}} = \frac{1}{128}\int d^2\theta\; d^2\bar\theta\, \frac{d^4p_1\, d^4p_2\, d^4p_3\, d^4p_4}{(2\pi)^{16}}\,\delta\Big(\sum p_i\Big)\,\mathcal{T}(1a,2b,3c,4d)\,\times \\ \times\int d^4k \int_0^\infty \prod_{i=1}^{4} d\alpha_i\, e^{-\alpha k^2}\, e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)}\,\times \\ \times\Big[\mathcal{A}(k,p_1,\ldots,p_4) + \mathcal{B}(k,p_1,\ldots,p_4)\Big]\,,\end{aligned} \tag{4.1}$$

where $\mathcal{T}(1a,2b,3c,4d)$ is defined in (2.15) and we have performed the shift

$$k \longrightarrow k - \frac{1}{\alpha}\Big[\alpha_1 p_1 - \alpha_3 p_4 + \alpha_4(p_1+p_2)\Big] \tag{4.2}$$

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in the loop-momentum integral. Moreover we have defined the symmetric function

$$
\begin{aligned}
f(p_1,p_2,p_3,p_4;\alpha_i) = {} & p_1^2\left[\alpha_1\alpha_2+\frac{1}{4}(\alpha_2\alpha_4+\alpha_1\alpha_3)\right]+p_2^2\left[\alpha_1\alpha_4+\frac{1}{4}(\alpha_2\alpha_4+\alpha_1\alpha_3)\right]+ \\
& +p_3^2\left[\alpha_3\alpha_4+\frac{1}{4}(\alpha_2\alpha_4+\alpha_1\alpha_3)\right]+ \\
& +p_4^2\left[\alpha_2\alpha_3+\frac{1}{4}(\alpha_2\alpha_4+\alpha_1\alpha_3)\right]+ \\
& +\frac{1}{2}\Big(p_1\cdot p_2+p_3\cdot p_4-p_1\cdot p_4-p_2\cdot p_3\Big)\,(\alpha_2\alpha_4-\alpha_1\alpha_3)- \\
& -\frac{1}{2}\,(p_1\cdot p_3+p_2\cdot p_4)\,(\alpha_2\alpha_4+\alpha_1\alpha_3)\,. \qquad (4.3)
\end{aligned}
$$

With $\mathcal{A}(k,p_1,\ldots,p_4)$ and $\mathcal{B}(k,p_1,\ldots,p_4)$ we have denoted the sum of all the contributions proportional to $\mathrm{Tr}(T^pT^q)\,\mathrm{Tr}(T^rT^s)$ and to $\mathrm{Tr}(T^p)\,\mathrm{Tr}(T^qT^rT^s)$, respectively. We list now separately the terms in $\mathcal{A}$ and $\mathcal{B}$. We have

$$
\begin{aligned}
\mathcal{A}(k,p_1,\ldots,p_4) = {} & P_1P_2T_3T_4+P_1T_2T_3P_4+P_1T_2P_3T_4+ \\
& +T_1T_2P_3P_4+T_1P_2P_3T_4+T_1P_2T_3P_4\,, \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
P_1P_2T_3T_4 \longrightarrow {} & \mathrm{Tr}(T^aT^b)\,\mathrm{Tr}(T^cT^d)\,e^{-\frac{i}{2}(p_1\times p_2-p_3\times p_4)}\times \\
& \times e^{-ik\times(p_1+p_2)}\,e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2-\alpha_3 p_3\times p_4]}\,, \\
P_1T_2T_3P_4 \longrightarrow {} & \mathrm{Tr}(T^aT^d)\,\mathrm{Tr}(T^bT^c)\,e^{-\frac{i}{2}(p_1\times p_4-p_2\times p_3)}\times \\
& \times e^{-ik\times(p_1+p_4)}\,e^{\frac{i}{\alpha}[(\alpha_1+\alpha_3+\alpha_4)p_1\times p_4-\alpha_4 p_2\times p_3]}\,, \\
P_1T_2P_3T_4 \longrightarrow {} & \mathrm{Tr}(T^aT^c)\,\mathrm{Tr}(T^bT^d)\,e^{-\frac{i}{2}(p_1\times p_3-p_2\times p_4)}\times \\
& \times e^{-ik\times(p_1+p_3)}\,e^{\frac{i}{\alpha}[(\alpha_1+\alpha_4)p_1\times p_3-(\alpha_3+\alpha_4)p_2\times p_4]}\,. \qquad (4.5)
\end{aligned}
$$

The additional three contributions $T_1T_2P_3P_4$, $T_1P_2P_3T_4$ and $T_1P_2T_3P_4$ are obtained from the corresponding terms in (4.5) by hermitean conjugation. In the same way we have

$$
\begin{aligned}
\mathcal{B}(k,p_1,\ldots,p_4) = {} & P_1P_2P_3T_4+P_1T_2P_3P_4+P_1P_2T_3P_4+T_1P_2P_3P_4+ \\
& +T_1T_2T_3P_4+T_1P_2T_3T_4+T_1T_2P_3T_4+P_1T_2T_3T_4\,, \qquad (4.6)
\end{aligned}
$$

where

$$
\begin{aligned}
P_1P_2P_3T_4 \longrightarrow {} & -\mathrm{Tr}(T^aT^bT^c)\,\mathrm{Tr}(T^d)e^{-\frac{i}{2}(p_1\times p_2+p_2\times p_3+p_1\times p_3)}\times \\
& \times e^{-ik\times(p_1+p_2+p_3)}\,e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2+\alpha_4 p_2\times p_3+(\alpha_1+\alpha_4)p_1\times p_3]}\,, \\
P_1T_2P_3P_4 \longrightarrow {} & -\mathrm{Tr}(T^cT^dT^a)\,\mathrm{Tr}(T^b)e^{-\frac{i}{2}(p_1\times p_3+p_3\times p_4+p_1\times p_4)}\times \\
& \times e^{-ik\times(p_1+p_3+p_4)}\,e^{\frac{i}{\alpha}[(\alpha_1+\alpha_4)p_1\times p_3+\alpha_3 p_3\times p_4+(\alpha_1+\alpha_3+\alpha_4)p_1\times p_4]}\,,
\end{aligned}
$$

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$$
\begin{aligned}
P_1P_2T_3P_4 \longrightarrow & - \mathrm{Tr}(T^dT^aT^b)\,\mathrm{Tr}(T^c) e^{-\frac{i}{2}(p_1\times p_2+p_2\times p_4+p_1\times p_4)} \times \\
& \times e^{-ik\times(p_1+p_2+p_4)}\, e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2+(\alpha_3+\alpha_4)p_2\times p_4+(\alpha_1+\alpha_3+\alpha_4)p_1\times p_4]}\,, \\
T_1P_2P_3P_4 \longrightarrow & - \mathrm{Tr}(T^bT^cT^d)\,\mathrm{Tr}(T^a) e^{-\frac{i}{2}(p_2\times p_3+p_3\times p_4+p_2\times p_4)} \times \\
& \times e^{-ik\times(p_2+p_3+p_4)}\, e^{\frac{i}{\alpha}[\alpha_4 p_2\times p_3+\alpha_3 p_3\times p_4+(\alpha_3+\alpha_4)p_2\times p_4]}\,.
\end{aligned}
\tag{4.7}
$$

As in the previous case the additional terms $T_1T_2T_3P_4$, $T_1P_2T_3T_4$, $T_1T_2P_3T_4$ and $P_1T_2T_3T_4$ are the hermitean conjugates of the contributions in (4.7).

Now in (4.1) the presence of the completely symmetric expression $\mathcal{T}(1a, 2b, 3c, 4d)$ allows to freely symmetrize the rest of the terms with respect to the exchanges $(1a) \leftrightarrow (2b) \leftrightarrow (3c) \leftrightarrow (4d)$. Operating in this way we rewrite

$$
\begin{aligned}
& e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)}\mathcal{A}(k,p_1,\ldots,p_4) \\
& \quad \longrightarrow \mathrm{Tr}(T^aT^b)\ \mathrm{Tr}(T^cT^d)\, e^{-\frac{i}{2}(p_1\times p_2-p_3\times p_4)}\, e^{-ik\times(p_1+p_2)} \times \\
& \qquad \times \Big[e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)} e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2-\alpha_3 p_3\times p_4]} + \\
& \qquad\quad + e^{-\frac{1}{\alpha}f(p_1,p_3,p_4,p_2;\alpha_i)} e^{\frac{i}{\alpha}[(\alpha_1+\alpha_3+\alpha_4)p_1\times p_2-\alpha_4 p_3\times p_4]} + \\
& \qquad\quad + e^{-\frac{1}{\alpha}f(p_1,p_3,p_2,p_4;\alpha_i)} e^{\frac{i}{\alpha}[(\alpha_1+\alpha_4)p_1\times p_2-(\alpha_3+\alpha_4)p_3\times p_4]}\Big] + \text{h.c.}
\end{aligned}
\tag{4.8}
$$

Similarly we obtain

$$
\begin{aligned}
& e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)}\mathcal{B}(k,p_1,\ldots,p_4) \\
& \quad \longrightarrow - \mathrm{Tr}(T^aT^bT^c)\ \mathrm{Tr}(T^d) e^{-\frac{i}{2}(p_1\times p_2+p_2\times p_3+p_1\times p_3)} e^{-ik\times(p_1+p_2+p_3)} \times \\
& \qquad \times \Big[e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)} e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2+\alpha_4 p_2\times p_3+(\alpha_1+\alpha_4)p_1\times p_3]} + \\
& \qquad\quad + e^{-\frac{1}{\alpha}f(p_1,p_4,p_2,p_3;\alpha_i)} e^{\frac{i}{\alpha}[(\alpha_1+\alpha_4)p_1\times p_2+\alpha_3 p_2\times p_3+(\alpha_1+\alpha_3+\alpha_4)p_1\times p_3]} + \\
& \qquad\quad + e^{-\frac{1}{\alpha}f(p_1,p_2,p_4,p_3;\alpha_i)} e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2+(\alpha_3+\alpha_4)p_2\times p_3+(\alpha_1+\alpha_3+\alpha_4)p_1\times p_3]} + \\
& \qquad\quad + e^{-\frac{1}{\alpha}f(p_4,p_1,p_2,p_3;\alpha_i)} e^{\frac{i}{\alpha}[\alpha_4 p_1\times p_2+\alpha_3 p_2\times p_3+(\alpha_3+\alpha_4)p_1\times p_3]}\Big] + \text{h.c.}
\end{aligned}
\tag{4.9}
$$

Substituting (4.8) and (4.9) in (4.1) we obtain the full result for the one-loop contributions to the four-point function of the $\mathcal{N} = 4$ non-commutative Yang-Mills theory

$$
\begin{aligned}
\Gamma_{\text{total}} = & \frac{1}{128}\int d^2\theta\, d^2\bar\theta\, \frac{d^4p_1\, d^4p_2\, d^4p_3\, d^4p_4}{(2\pi)^{16}}\, \delta\left(\sum p_i\right) \times \\
& \times \Big[\mathbf{W}^{\alpha a}(p_1)\mathbf{W}^b_\alpha(p_2)\bar{\mathbf{W}}^{\dot\alpha c}(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4) + \\
& \quad + \mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\bar{\mathbf{W}}^c_{\dot\alpha}(p_3)\mathbf{W}^d_\alpha(p_4) - \\
& \quad - \mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\mathbf{W}^c_\alpha(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4) + \text{h.c.}\Big] \times \\
& \times \int d^4k \int_0^\infty \prod_{i=1}^4 d\alpha_i\, e^{-\alpha k^2} \times \\
& \quad \times \Big\{N\,\mathrm{Tr}(T^aT^bT^cT^d) e^{\frac{i}{2}(p_2\times p_1+p_1\times p_4+p_2\times p_4)} e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)} +
\end{aligned}
$$

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$$
\begin{aligned}
&+\ \mathrm{Tr}(T^aT^b)\ \mathrm{Tr}(T^cT^d)\, e^{-\frac{i}{2}(p_1\times p_2-p_3\times p_4)}\, e^{-ik\times(p_1+p_2)}\times\\
&\quad\times\Big[e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)}e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2-\alpha_3 p_3\times p_4]}+\\
&\qquad+\ e^{-\frac{1}{\alpha}f(p_1,p_3,p_4,p_2;\alpha_i)}e^{\frac{i}{\alpha}[(\alpha_1+\alpha_3+\alpha_4)p_1\times p_2-\alpha_4 p_3\times p_4]}+\\
&\qquad+e^{-\frac{1}{\alpha}f(p_1,p_3,p_2,p_4;\alpha_i)}e^{\frac{i}{\alpha}[(\alpha_1+\alpha_4)p_1\times p_2-(\alpha_3+\alpha_4)p_3\times p_4]}\Big]-\\
&-\ \mathrm{Tr}(T^aT^bT^c)\ \mathrm{Tr}(T^d)e^{-\frac{i}{2}(p_1\times p_2+p_2\times p_3+p_1\times p_3)}\ e^{-ik\times(p_1+p_2+p_3)}\times\\
&\quad\times\Big[e^{-\frac{1}{\alpha}f(p_1,p_2,p_3,p_4;\alpha_i)}e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2+\alpha_4 p_2\times p_3+(\alpha_1+\alpha_4)p_1\times p_3]}+\\
&\qquad+e^{-\frac{1}{\alpha}f(p_1,p_4,p_2,p_3;\alpha_i)}e^{\frac{i}{\alpha}[(\alpha_1+\alpha_4)p_1\times p_2+\alpha_3 p_2\times p_3+(\alpha_1+\alpha_3+\alpha_4)p_1\times p_3]}+\\
&\qquad+e^{-\frac{1}{\alpha}f(p_1,p_2,p_4,p_3;\alpha_i)}e^{\frac{i}{\alpha}[\alpha_1 p_1\times p_2+(\alpha_3+\alpha_4)p_2\times p_3+(\alpha_1+\alpha_3+\alpha_4)p_1\times p_3]}+\\
&\qquad+e^{-\frac{1}{\alpha}f(p_4,p_1,p_2,p_3;\alpha_i)}e^{\frac{i}{\alpha}[\alpha_4 p_1\times p_2+\alpha_3 p_2\times p_3+(\alpha_3+\alpha_4)p_1\times p_3]}\Big]+\mathrm{h.c.}\Big\}\,.
\end{aligned}
\tag{4.10}
$$

Using (2.16) one can extract the purely bosonic terms contained in (4.10). The above result can be easily compared with corresponding expressions obtained from the field theory limit of open string amplitudes in the presence of a constant B-field. To this end it is sufficient to express the Schwinger parameters α_i, $i=1,\ldots,4$, in terms of variables λ, ξ_i, $i=1,2,3$, that appear naturally in string loop calculations, e.g. with the ordering $\xi_1>\xi_2>\xi_3$

$$
\begin{aligned}
&\lambda\equiv\alpha+\alpha_1+\alpha_2+\alpha_3+\alpha_4\,,\\
&\alpha_1=\lambda(\xi_1-\xi_2)\,,\qquad \alpha_2=\lambda(1-\xi_1)\,,\\
&\alpha_3=\lambda(\xi_2-\xi_3)\,,\qquad \alpha_4=\lambda\xi_3\,.
\end{aligned}
\tag{4.11}
$$

In section 5 we make use of such changes of the integration variables in order to study the four-point function in the low-energy limit. In this case the integrations on the λ and ξ_i variables can be performed exactly. Now we turn to this computation.

5. The low-energy four-point amplitude

We study the amplitude (4.10) in the low-energy approximation $p_i\cdot p_j$ small, with $p_i\times p_j$ finite. In order not to worry about IR divergences we use a mass regulator. We introduce the change of variables in (4.11) and perform the explicit integration on the loop momentum k. We obtain

$$
\begin{aligned}
\Gamma_{\mathrm{l.e.}}=\frac{\pi^2}{128}\int d^2\theta\, d^2\bar\theta\,&\frac{d^4p_1\,d^4p_2\,d^4p_3\,d^4p_4}{(2\pi)^{16}}\,\delta\left(\sum p_i\right)\times\\
&\times\Big[\mathbf{W}^{\alpha a}(p_1)\mathbf{W}^b_\alpha(p_2)\bar{\mathbf{W}}^{\dot\alpha c}(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4)+\\
&\quad+\ \mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\bar{\mathbf{W}}^c_{\dot\alpha}(p_3)\mathbf{W}^d_\alpha(p_4)-\\
&\quad-\ \mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\mathbf{W}^c_\alpha(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4)+\mathrm{h.c.}\Big]\times
\end{aligned}
$$

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$$
\begin{aligned}
&\times \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \int_0^\infty d\lambda\,\lambda\,\times \\
&\quad\times \Big\{ e^{-\lambda m^2}\, N\ \mathrm{Tr}(T^aT^bT^cT^d)\, e^{\frac{i}{2}(p_2\times p_1+p_1\times p_4+p_2\times p_4)} + \\
&\qquad + e^{-\frac{1}{4\lambda}(p_1+p_2)\circ(p_1+p_2)-\lambda m^2}\, e^{-\frac{i}{2}(p_1\times p_2-p_3\times p_4)}\ \mathrm{Tr}(T^aT^b)\ \mathrm{Tr}(T^cT^d)\times \\
&\qquad \times \Big[e^{i[(\xi_1-\xi_2)p_1\times p_2-(\xi_2-\xi_3)p_3\times p_4]} + e^{i[\xi_1 p_1\times p_2-\xi_3 p_3\times p_4]} + \\
&\qquad\quad + e^{i[(\xi_1-\xi_2+\xi_3)p_1\times p_2-\xi_2 p_3\times p_4]}\Big] - e^{-\frac{1}{4\lambda}(p_1+p_2+p_3)\circ(p_1+p_2+p_3)-\lambda m^2}\times \\
&\qquad \times e^{-\frac{i}{2}(p_1\times p_2+p_2\times p_3+p_1\times p_3)}\ \mathrm{Tr}(T^aT^bT^c)\,\mathrm{Tr}(T^d)\times \\
&\qquad \times \Big[e^{i[(\xi_1-\xi_2)p_1\times p_2+\xi_3 p_2\times p_3+(\xi_1-\xi_2+\xi_3)p_1\times p_3]} + \\
&\qquad\quad + e^{i[(\xi_1-\xi_2+\xi_3)p_1\times p_2+(\xi_2-\xi_3)p_2\times p_3+\xi_1 p_1\times p_3]} + \\
&\qquad\quad + e^{i[(\xi_1-\xi_2)p_1\times p_2+\xi_2 p_2\times p_3+\xi_1 p_1\times p_3]} + \\
&\qquad\quad + e^{i[\xi_3 p_1\times p_2+(\xi_2-\xi_3)p_2\times p_3+\xi_2 p_1\times p_3]}\Big] + \mathrm{h.c.}\Big\}\,, \qquad (5.1)
\end{aligned}
$$

where we have defined $p\circ p = p_\mu \Theta^{\mu\nu}\Theta_{\rho\nu}p^\rho$. Finally we perform the integration on the variables λ and ξ_i. Exploiting all the symmetries of the integrand the complete result for the low-energy planar and non-planar contributions to the four-point function can be written as

$$
\begin{aligned}
\Gamma_{\mathrm{l.e.}} = &\frac{\pi^2}{128}\int d^2\theta\, d^2\bar\theta\, \frac{d^4p_1\, d^4p_2\, d^4p_3\, d^4p_4}{(2\pi)^{16}}\,\delta\left(\sum p_i\right)\times \\
&\times \Big[\mathbf{W}^{\alpha a}(p_1)\mathbf{W}^b_\alpha(p_2)\bar{\mathbf{W}}^{\dot\alpha c}(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4) + \\
&\quad + \mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\bar{\mathbf{W}}^c_{\dot\alpha}(p_3)\mathbf{W}^d_\alpha(p_4) - \\
&\quad - \mathbf{W}^{\alpha a}(p_1)\bar{\mathbf{W}}^{\dot\alpha b}(p_2)\mathbf{W}^c_\alpha(p_3)\bar{\mathbf{W}}^d_{\dot\alpha}(p_4) + \mathrm{h.c.}\Big]\times \\
&\times \Big\{\frac{1}{6m^4}\, N\ \mathrm{Tr}(T^aT^bT^cT^d)\, e^{\frac{i}{2}(p_2\times p_1+p_1\times p_4+p_2\times p_4)} + \\
&\quad + \frac{1}{m^2}\,\frac{(p_1+p_2)\circ(p_1+p_2)}{4}\,\frac{\sin(p_1\times p_2/2)}{p_1\times p_2/2}\,\frac{\sin(p_3\times p_4/2)}{p_3\times p_4/2}\times \\
&\qquad \times K_2\big(m\sqrt{(p_1+p_2)\circ(p_1+p_2)}\big)\ \mathrm{Tr}(T^aT^b)\ \mathrm{Tr}(T^cT^d) - \\
&\quad - \frac{1}{m^2}\,\frac{(p_1+p_2+p_3)\circ(p_1+p_2+p_3)}{2}\,\frac{e^{-\frac{i}{2}(p_1\times p_2+p_2\times p_3+p_1\times p_3)}}{(p_1\times p_4)(p_3\times p_4)}\times \\
&\qquad \times K_2\big(m\sqrt{(p_1+p_2+p_3)\circ(p_1+p_2+p_3)}\big)\ \mathrm{Tr}(T^aT^bT^c)\,\mathrm{Tr}(T^d) + \\
&\quad + \mathrm{h.c.}\Big\} \qquad (5.2)
\end{aligned}
$$

Again using (2.16) we can extract the purely bosonic contributions in terms of the $F_{\mu\nu}$ electromagnetic field strengths. The resulting expression coincides with the one-loop field theory limit obtained in [10] from open string amplitudes. We note that, as in [10] only the terms from planar diagrams contain phase factors depending on the momenta in such a way to reconstruct the $*$-product between the superfield strengths. The remaining two structures which arise from non-planar contributions have somewhat modified multiplication rules, which might endanger the gauge invariance of the four-point amplitude.

6. Conclusions

The classical action in (2.2) is invariant under non-linear gauge transformations, which are just the generalization to the non-commutative case of the standard ones [6, 11]

$$\begin{aligned} e^V &\longrightarrow e_*^{i\bar{\Lambda}} * e_*^V * e_*^{-i\Lambda}\,, \\ \Phi &\longrightarrow e_*^{i\Lambda} * \Phi * e_*^{-i\Lambda}\,, \qquad \bar{\Phi} \longrightarrow e_*^{i\bar{\Lambda}} * \bar{\Phi} * e_*^{-i\bar{\Lambda}} \end{aligned} \tag{6.1}$$

with a gauge parameter Λ which is a chiral superfield.

In the commutative case the background covariant quantization is extremely advantageous for gauge theories: the quantum-background splitting introduces separate gauge transformations for the quantum and the background fields. The gauge-fixing procedure breaks the quantum gauge invariance, while maintaining explicit the background gauge invariance. Thus the perturbative effective action is expressed in terms of covariant derivatives and field strengths, with no need of implementing gauge invariance through Ward identities.

When the non-commutative theories are constructed via the introduction of the $*$-multiplication and the gauge transformations are modified according to (6.1) the classical invariance is guaranteed. Use of the background field method allows to obtain perturbative results in terms again of covariant objects, but as we have found in (5.2) the $*$-product is not reproduced and new multiplication rules appear, i.e. $*'$- and $*_3$-products [10, 16, 17]. Gauge invariance is not anymore manifest and seems to be lost [9, 10]. One might worry that something went wrong in applying standard perturbation theory techniques to a theory which is non commutative and non local. Or else should one expect to recover gauge invariance only at the level of the S-matrix?

It is worthwhile to investigate further. The generalized $*$-products also appear in the solution of the Seiberg-Witten equation [2, 16, 17] and in the expansion of gauge invariant operators introduced in [18, 19]. This seems to be an indication that one is on the right track, even if not quite. It has been shown that open Wilson lines play an important role in the construction of gauge invariant operators [18, 19] and it is conceivable that they might be equally important at the level of the effective action. Following this line of thoughts it could be interesting to look for modifications of the Wick expansion prescription or for a new mechanism that would allow to implement the open Wilson lines directly in the action.

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Note added in proof. After the completion of this work in refs. [20] the gauge invariance of the one-loop effective action has been proven by taking into account contributions from higher-order point functions.

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References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162]; M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[2] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[3] N. Seiberg, L. Susskind, N. Toumbas, *J. High Energy Phys.* **06** (2000) 021 [hep-th/000540]; R. Gopakumar, J. Maldacena, S. Minwalla and A. Strominger, *S-duality and noncommutative gauge theory*, *J. High Energy Phys.* **06** (2000) 036 [hep-th/0005048];
R. Gopakumar, S. Minwalla, N. Seiberg and A. Strominger, *OM theory in diverse dimensions*, *J. High Energy Phys.* **08** (2000) 008 [hep-th/0006062];
E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, *Critical fields on the M5-brane and noncommutative open strings*, *Phys. Lett.* **B 492** (2000) 193 [hep-th/0006112].

[4] O. Andreev and H. Dorn, *Diagrams of noncommutative* Φ^3 *theory from string theory*, *Nucl. Phys.* **B 583** (2000) 145 [hep-th/0003113];
A. Bilal, C.-S. Chu and R. Russo, *String theory and noncommutative field theories at one loop*, *Nucl. Phys.* **B 582** (2000) 65 [hep-th/0003180];
C.-S. Chu, R. Russo and S. Sciuto, *Multiloop string amplitudes with B-field and noncommutative QFT*, *Nucl. Phys.* **B 585** (2000) 193 [hep-th/0004183].

[5] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53;
S. Minwalla, M.V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, hep-th/9912072;
A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the noncommutative gauge theories*, hep-th/0002075;
M.V. Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, *J. High Energy Phys.* **03** (2000) 035 [hep-th/0002186];

JHEP01(2001)024

A. Armoni, *Comments on perturbative dynamics of non-commutative Yang-Mills theory*, *Nucl. Phys.* **B 593** (2001) 229 [hep-th/0005208];
I. Chepelev and R. Roiban, *Renormalization of quantum field theories on noncommutative R^d, 1. Scalars*, *J. High Energy Phys.* **05** (2000) 037 [hep-th/9911098]; *Convergence theorem for non-commutative Feynman graphs and renormalization*, hep-th/0008090.

[6] S. Ferrara and M.A. Lledo, *Some aspects of deformations of supersymmetric field theories*, *J. High Energy Phys.* **05** (2000) 008 [hep-th/0002084].

[7] C.-S. Chu and F. Zamora, *Manifest supersymmetry in non-commutative geometry*, *J. High Energy Phys.* **02** (2000) 022 [hep-th/9912153];
S. Terashima, *A note on superfields and noncommutative geometry*, *Phys. Lett.* **B 482** (2000) 276 [hep-th/0002119];
N. Grandi, R.L. Pakman and F.A. Schaposnik, *Supersymmetric Dirac-Born-Infeld theory in noncommutative space*, *Nucl. Phys.* **B 588** (2000) 508 [hep-th/0004104];
M.T. Grisaru and S. Penati, *Noncommutative supersymmetric gauge anomaly*, hep-th/0010177.

[8] A.A. Bichl, I.M. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, *The superfield formalism applied to the noncommutative Wess-Zumino model*, *J. High Energy Phys.* **10** (2000) 046 [hep-th/0007050].

[9] D. Zanon, *Noncommutative perturbation in superspace*, hep-th/0009196.

[10] H. Liu and J. Michelson, *$*$-Trek: the one loop $N = 4$ noncommutative SYM action*, hep-th/0008205.

[11] S.J. Gates, M.T. Grisaru, M. Rocek and W. Siegel, *Superspace*, Benjamin-Cummings, Reading MA, 1983.

[12] M.T. Grisaru and W. Siegel, *Supergraphity, 2. Manifestly covariant rules and higher loop finiteness*, *Nucl. Phys.* **B 201** (1982) 292.

[13] M.T. Grisaru and D. Zanon, *Covariant supergraphs, 1. Yang-Mills theory*, *Nucl. Phys.* **B 252** (1985) 578.

[14] M.T. Grisaru and W. Siegel, *The one loop four particle s matrix in extended supergravity*, *Phys. Lett.* **B 110** (1982) 49.

[15] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, 1987.

[16] M.R. Garousi, *Non-commutative world-volume interactions on D-branes and Dirac-Born-Infeld action*, *Nucl. Phys.* **B 579** (2000) 209 [hep-th/9909214].

[17] T. Mehen and M.B. Wise, *Generalized $*$-products, Wilson lines and the solution of the Seiberg-Witten equations*, *J. High Energy Phys.* **12** (2000) 008 [hep-th/0010204].

JHEP01(2001)024

[18] S.R. Das and S.-J. Rey, *Open Wilson lines in noncommutative gauge theory and tomography of holographic dual supergravity*, *Nucl. Phys.* **B 590** (2000) 453 [hep-th/0008042];
D.J. Gross, A. Hashimoto and N. Itzhaki, *Observables of non-commutative gauge theories*, hep-th/0008075.

[19] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, *Nucl. Phys.* **B 573** (2000) 573 [hep-th/9910004].

[20] H. Liu, *$*$-Trek II: $*_n$ operations, open Wilson lines and the Seiberg-Witten map*, hep-th/0011125;
M. Pernici, A. Santambrogio and D. Zanon, *The one-loop effective action of noncommutative $N = 4$ super Yang-Mills is gauge invariant*, hep-th/0011140.

JHEP01(2001)024